

## Problem 2.17

[Difficulty: 4]

**2.17** Verify that  $x_p = -a \sin(\omega t)$ ,  $y_p = a \cos(\omega t)$  is the equation for the pathlines of particles for the flow field of Problem 2.12. Find the frequency of motion  $\omega$  as a function of the amplitude of motion,  $a$ , and  $K$ . Verify that  $x_p = -a \sin(\omega t)$ ,  $y_p = a \cos(\omega t)$  is also the equation for the pathlines of particles for the flow field of Problem 2.11, except that  $\omega$  is now a function of  $M$ . Plot typical pathlines for both flow fields and discuss the difference.

**Given:** Pathlines of particles

**Find:** Conditions that make them satisfy Problem 2.10 flow field; Also Problem 2.11 flow field; Plot pathlines

**Solution:**

The given pathlines are

$$x_p = -a \sin(\omega t) \quad y_p = a \cos(\omega t)$$

The velocity field of Problem 2.12 is

$$u = -\frac{K \cdot y}{2 \cdot \pi \cdot (x^2 + y^2)} \quad v = \frac{K \cdot x}{2 \cdot \pi \cdot (x^2 + y^2)}$$

If the pathlines are correct we should be able to substitute  $x_p$  and  $y_p$  into the velocity field to find the velocity as a function of time:

$$u = -\frac{K \cdot y}{2 \cdot \pi \cdot (x^2 + y^2)} = -\frac{K \cdot a \cdot \cos(\omega t)}{2 \cdot \pi \cdot (a^2 \cdot \sin^2(\omega t) + a^2 \cdot \cos^2(\omega t))} = -\frac{K \cdot \cos(\omega t)}{2 \cdot \pi \cdot a} \quad (1)$$

$$v = \frac{K \cdot x}{2 \cdot \pi \cdot (x^2 + y^2)} = -\frac{K \cdot (-a \cdot \sin(\omega t))}{2 \cdot \pi \cdot (a^2 \cdot \sin^2(\omega t) + a^2 \cdot \cos^2(\omega t))} = -\frac{K \cdot \sin(\omega t)}{2 \cdot \pi \cdot a} \quad (2)$$

We should also be able to find the velocity field as a function of time from the pathline equations (Eq. 2.9):

$$\frac{dx_p}{dt} = u \quad \frac{dx_p}{dt} = v \quad (2.9)$$

$$u = \frac{dx_p}{dt} = -a \cdot \omega \cdot \cos(\omega t) \quad v = \frac{dy_p}{dt} = -a \cdot \omega \cdot \sin(\omega t) \quad (3)$$

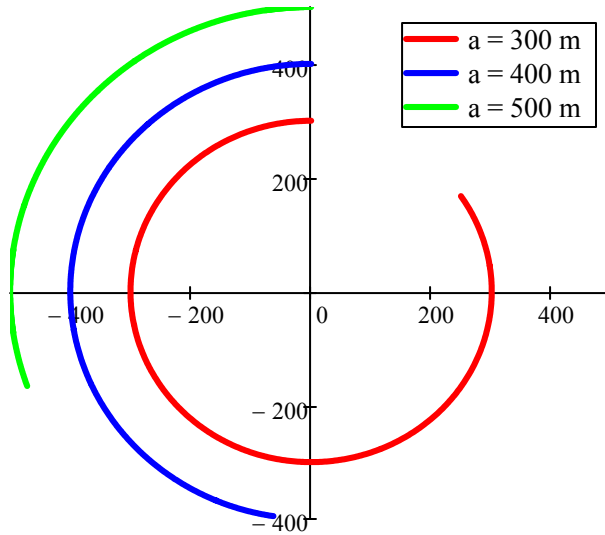
Comparing Eqs. 1, 2 and 3

$$u = -a \cdot \omega \cdot \cos(\omega t) = -\frac{K \cdot \cos(\omega t)}{2 \cdot \pi \cdot a} \quad v = -a \cdot \omega \cdot \sin(\omega t) = -\frac{K \cdot \sin(\omega t)}{2 \cdot \pi \cdot a}$$

Hence we see that

$$a \cdot \omega = \frac{K}{2 \cdot \pi \cdot a} \quad \text{or} \quad \omega = \frac{K}{2 \cdot \pi \cdot a^2} \quad \text{for the pathlines to be correct.}$$

The pathlines are



To plot this in Excel, compute  $x_p$  and  $y_p$  for  $t$  ranging from 0 to 60 s, with  $\omega$  given by the above formula. Plot  $y_p$  versus  $x_p$ . Note that outer particles travel much slower!

This is the free vortex flow discussed in Example 5.6

The velocity field of Problem 2.11 is

$$u = -\frac{M \cdot y}{2 \cdot \pi} \quad v = \frac{M \cdot x}{2 \cdot \pi}$$

If the pathlines are correct we should be able to substitute  $x_p$  and  $y_p$  into the velocity field to find the velocity as a function of time:

$$u = -\frac{M \cdot y}{2 \cdot \pi} = -\frac{M \cdot (a \cdot \cos(\omega \cdot t))}{2 \cdot \pi} = -\frac{M \cdot a \cdot \cos(\omega \cdot t)}{2 \cdot \pi} \quad (4)$$

$$v = \frac{M \cdot x}{2 \cdot \pi} = \frac{M \cdot (-a \cdot \sin(\omega \cdot t))}{2 \cdot \pi} = -\frac{M \cdot a \cdot \sin(\omega \cdot t)}{2 \cdot \pi} \quad (5)$$

Recall that

$$u = \frac{dx_p}{dt} = -a \cdot \omega \cdot \cos(\omega \cdot t) \quad v = \frac{dy_p}{dt} = -a \cdot \omega \cdot \sin(\omega \cdot t) \quad (3)$$

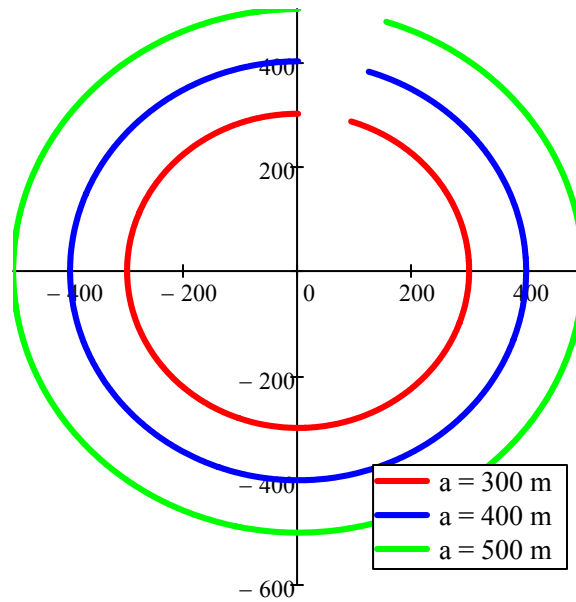
Comparing Eqs. 1, 4 and 5

$$u = -a \cdot \omega \cdot \cos(\omega \cdot t) = -\frac{M \cdot a \cdot \cos(\omega \cdot t)}{2 \cdot \pi} \quad v = -a \cdot \omega \cdot \sin(\omega \cdot t) = -\frac{M \cdot a \cdot \sin(\omega \cdot t)}{2 \cdot \pi}$$

Hence we see that

$$\omega = \frac{M}{2 \cdot \pi} \quad \text{for the pathlines to be correct.}$$

## The pathlines



To plot this in Excel, compute  $x_p$  and  $y_p$  for  $t$  ranging from 0 to 75 s, with  $\omega$  given by the above formula. Plot  $y_p$  versus  $x_p$ . Note that outer particles travel faster!

This is the forced vortex flow discussed in Example 5.6

Note that this is rigid body rotation!